Quantum Field Theory and Computational Fluid Dynamics

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Based on research with F. Bezrukov, V. Rubakov and P. Tinyakov
Baryon number violation
in high-energy electroweak processes

The standard model of electroweak interactions is formulated in terms of bosonic fields describing $\gamma$, $W^\pm$, $Z$ and the Higgs particle, and fields describing fermionic particles.

The configurations of the bosonic fields can be classified into inequivalent topological sectors.

Generally in electroweak processes, as in all known particle interactions, net baryon number $N_B = n_B - n_\bar{B}$ is conserved. By a subtle quantum-mechanical effect (renormalization anomaly) $N_B$ changes if the process changes the topology of the fields.
Field topology

We will consider field configurations with spherical symmetry. The bosonic sector is described by two complex fields and one real field: $\chi(r, t), \bar{\chi}(r, t), \phi(r, t), \bar{\phi}(r, t), a(r, t)$.

**Non-trivial topology** occurs because in the state of lowest energy (vacuum) $|\chi| = \chi_0 > 0, \quad |\phi| = \phi_0 > 0$.

Regularity demands $\chi(r = 0, t) = \chi(r = \infty, t) = \text{const}$: vacuum states are characterized by their topological winding number $Q$. 
$\chi(r,t)$:

$\Delta g = 0$

$\Delta g = 1$

In a transition where $\Delta g \neq 0$ also $\Delta N_B \neq 0$ ($\Delta N_B = 3\Delta g$)
Sectors with different $Q$ are separated by an energy barrier:

The **sphaleron** is an unstable solution lying on top of the barrier. For $E > E_{sph}$, $Q$ can change in a classical evolution. For $E < E_{sph}$, $Q$ can change only through quantum mechanical tunneling.
Can a two particle collision induce $\Delta Q \neq 0$ with appreciable probability, and, if so, at what energy?

Study processes which change topology with small $N_{\text{part}}$ in the initial state. Use semiclassical methods, solve eqns. of motion with:

- $t \ll 0$ imploding wave, in the linear regime
- $t \approx 0$ nonlinear evolution, topology changes (?)
- $t \gg 0$ exploding wave, in the linear regime.
Energy vs. incoming particle number

For $t \ll 0$ or $t \gg 0$ expand in normal modes $a(k)$:

$$E = \frac{1}{g^2} \int dk \omega(k) |a(k)|^2, \quad N = \frac{1}{g^2} \int dk |a(k)|^2$$

In A topology changing processes are classically allowed, in B they can only occur through tunneling.
Semiclassical approach to tunneling

Saddle point approximation leads to solving the evolution equations along a complex time contour:

Impose initial boundary conditions that project over definite energy and incoming particle number.
Computational method

Discretize the Lagrangian keeping exact gauge invariance.

Solve the equations of motion globally (elliptic equations cannot be evolved) at all space-time nodes, i.e. solve $N_r \times N_t$ ($\times 5$ complex) non-linear equations:

- start from an approximate solution;

- use Newton-Raphson method

$$e_i(\phi_j) = 0 \quad i = (r,t)$$

$$\rightarrow \sum_i \frac{\partial e_i}{\partial \phi_j} \delta \phi_j + e(\phi_j,0) = 0$$

this is a set of $N_r \times N_t$ ($\times 5$ complex) linear equations.

$\phi_t$ is coupled only to $\phi_{t+1}$ and $\phi_{t-1}$: eliminate alternate time slices, which requires full $5N_r \times 5N_r$ matrix manipulations, and reduce to boundary conditions for fields at initial and final times.
Configuration for $T/2 = 1$ and $\theta = 3$. $E/E_{\text{sph}} = 0.86$, $N = 0.48$.

Rotate (Local). Animate (Local).
Lines of $F(\varepsilon, \nu) = \log(\text{suppression factor}) = \text{const.}$

All values are normalized: $\varepsilon_{\text{sph}} = \nu_{\text{sph}} = 1$

$F(\varepsilon = 0, \nu = 0) = 1$
Dependence of suppression factor $F$ on $\nu$ for different energies. Extrapolation to $\nu = 0$ gives large suppression for all energies.
Conclusions

Research is still in progress, larger grids would allow for better linearization, singularity structure must be understood better.

The study opens new dimensions for the application of computational methods to quantum field theory.

It illustrates the power of coupling analytical methods with advanced computational techniques.
Configuration for $T/2 = 1$ and $\theta = 3.35$. $E/E_{\text{sph}} = 1.04$, $\nu = 0.56$.

Rotate (Local). Animate (Local).

Bounces back . . .
Consider the inclusive multiparticle probability of tunneling from a “microcanonical” state with energy $E$ and number of particles $N$:

$$
\sigma(E, N) = \sum_{i, f} |\langle f | \hat{S} \hat{P}_E \hat{P}_N | i \rangle|^2,
$$

where sum is performed over all states $|i\rangle$ and $|f\rangle$ near different vacua, and $\hat{P}_E$, $\hat{P}_N$ are projection operators onto subspaces of fixed energy and number of particles.

Writing a functional integral representation and making use of saddle point approximation one can arrive to the following “simple prescription” (Rubakov, Tinyakov, 92, Rubakov, Son, Tinyakov, 92):
Boundary value problem:

\[ \sigma(E, N) \sim \exp \left\{ -\frac{2\pi}{\alpha_W} F(\varepsilon, \nu) \right\}, \]

\[-\frac{2\pi}{\alpha_W} F(\varepsilon, \nu) = N\theta + ET + 2 \text{Re}[iS_{ABCD}(\varphi)] + \text{Re} B_i\]

where: \( \gamma = e^{-\theta} \),

\[ B_i = \frac{1}{2} \int dk (f_k f_{-k} e^{-2i\omega_k t_i} - g_k^* g_{-k}^* e^{2i\omega_k t_i}) \]
Fields satisfy the boundary value problem specified on the contour:

\[ \frac{\delta S}{\delta \varphi} = 0 \quad \text{and} \quad \begin{cases} 
\text{Im } \varphi(0, x) = \text{Im } \varphi(0, x) = 0, \\
 f_k = e^{-\theta} g_k, 
\end{cases} \]

\[ E = \int dk \omega_k f_k g_k^*, \quad N = \int dk f_k g_k^*. \]

\[ \varphi(x) \bigg|_{t_i} = \int \frac{dk}{(2\pi)^{3/2}} \frac{(f_k e^{-i\omega_k t + ikx} + g_k^* e^{i\omega_k t - ikx})}{\sqrt{2\omega_k}} \]
How to apply this to particle collisions?

One can easily obtain results with all these instanton–like solutions valid in semiclassical limit corresponding to states with large number of particles $N$:

$$\sigma(E, N) \sim \exp \left\{ -\frac{2\pi}{\alpha} F(\varepsilon, \nu) \right\}$$

where $\varepsilon = E/E_{\text{sph}}$, $\nu = N\alpha$, and $\alpha$ is a small parameter (coupling constant).

But—initial state in high energy collisions is generally not semiclassical: it contains a few (usually 2) particles.

Solution—take limit of $\nu \to 0$. 

The model

$SU(2)$ gauge theory with Higgs doublet

\[ S = \int dx^4 \left\{ -\frac{1}{2} \text{Tr} F_{\mu\nu}F^{\mu\nu} + (D_\mu \Phi)\dagger D^\mu \Phi - \lambda (\Phi\dagger \Phi - 1)^2 \right\} \]

- Corresponds to the Higgs sector of the Standard Model (with $\theta_W = 0$).

Note: In calculations $\lambda = 0.125$, so $M_H = M_W$.

4–dimensional—hardly manageable numerically. One may consider only spherically symmetric configurations and arrive with an effective 2–dimensional model:
1+1–dimensional effective model

\[ S = 4\pi \int dt \int_0^\infty dr \left[ -\frac{1}{4} r^2 f_{\mu\nu} f^{\mu\nu} + (D_\mu \chi)^* D^\mu \chi + r^2 (D_\mu \phi)^* D^\mu \phi \\
- \frac{1}{2r^2} (|\chi|^2 - 1)^2 - \frac{1}{2} (|\chi|^2 + 1)|\phi|^2 \\
- \text{Re}(i \chi^* \phi^2) - \lambda r^2 (|\phi|^2 - 1)^2 \right] \]

Where \( f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \) and \( \chi \) corresponds to gauge field of the original model, and \( \phi \)—to the original Higgs field.

This model retains all important features of the full model:

- Topological structure
- Linearizes at initial and final time going to infinity
Note: In $a_0 = 0$ gauge, it has 5 real fields and gauge invariance for time independent transformations.

Topological transition in terms of fields $\phi$ and $\chi$. 